I don't guarantee that these solutions are error free!

Derivation of formula for $\sum_{i=1}^{n} i^2$

$\sum_{i=1}^{n}(i+1)^3=\sum_{i=2}^{n+1}i^3=\sum_{i=1}^{n}i^3+(n+1)^3-1^3$ (from property of summation) (1)

but: $\sum_{i=1}^{n}(i+1)^3=\sum_{i=1}^{n}\left(i^3+3i^2+3i+1\right)$ (2)

from RHS of (2) and substituting $\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$ and $\sum_{i=1}^{n}1=n$:

$\sum_{i=1}^{n}i^3+3\sum_{i=1}^{n}i^2+3\sum_{i=1}^{n}i+\sum_{i=1}^{n}1=\sum_{i=1}^{n}i^3+3\sum_{i=1}^{n}i^2+3\frac{n(n+1)}{2}+n$ (3)

equating RHS of (3) and RHS of (1):

$$\cancel{\sum_{i=1}^{n}i^3}+3\sum_{i=1}^{n}i^2+3\frac{n(n+1)}{2}+n=\cancel{\sum_{i=1}^{n}i^3}+(n+1)^3-1$$

after rearranging:

$$\sum_{i=1}^{n}i^2=\frac{1}{3}\left[(n+1)^3-1-n-3\frac{n(n+1)}{2}\right]$$

$$=\frac{1}{3}\left[n^3+3n^2+3n+\cancel{1}-\cancel{1}-n-\frac{3}{2}n^2-\frac{3n}{2}\right]$$

$$=\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}=\frac{2n^3+3n^2+n}{6}=\frac{n(2n^2+3n+1)}{6}=\frac{n(n+1)(2n+1)}{6}$$

Thus: $\sum_{i=1}^{n}i^2=\frac{n(n+1)(2n+1)}{6}$

Derivation of formula for $\sum_{i=1}^{n} i^3$

$$\sum_{i=1}^{n}(i+1)^4 = \sum_{i=2}^{n+1} i^4 = \sum_{i=1}^{n} i^4 + (n+1)^4 - 1^4 \text{ (from property of summation)} \qquad (1)$$

but: $$\sum_{i=1}^{n}(i+1)^4 = \sum_{i=1}^{n}\left(i^4 + 4i^3 + 6i^2 + 4i + 1\right) \qquad (2)$$

from RHS of (2) and substituting $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} 1 = n$:

$$\sum_{i=1}^{n} i^4 + 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$= \sum_{i=1}^{n} i^4 + 4\sum_{i=1}^{n} i^3 + 6\,\frac{n(n+1)(2n+1)}{6} + 4\,\frac{n(n+1)}{2} + n \qquad (3)$$

equating RHS of (3) and RHS of (1):

$$\cancel{\sum_{i=1}^{n} i^4} + 4\sum_{i=1}^{n} i^3 + 6\,\frac{n(n+1)(2n+1)}{6} + 4\,\frac{n(n+1)}{2} + n = \cancel{\sum_{i=1}^{n} i^4} + (n+1)^4 - 1^4$$

after rearranging:

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}\left[(n+1)^4 - 1 - n - 2n(n+1) - n(n+1)(2n+1)\right]$$

$$= \frac{1}{4}\left[\left(n^4 + 4n^3 + 6n^2 + 4n + \cancel{1}\right) - \cancel{1} - n - \left(2n^2 + 2n\right) - \left(2n^3 + 3n^2 + n\right)\right]$$

$$= \frac{1}{4}\left[n^4 + 4n^3 + 6n^2 + 4n - n - 2n^2 - 2n - 2n^3 - 3n^2 - n\right]$$

$$= \frac{1}{4}\left[n^4 + 2n^3 + n^2\right] = \frac{n^2}{4}\left[n^2 + 2n + 1\right] = \frac{n^2(n+1)^2}{4}$$

Thus: $$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Derivation of a general formula for $\sum_{i=1}^{n} i^k$, where $k \in \mathbb{N}$

In general we see that to find the formula for power ***k*** we need to use $\sum_{i=1}^{n}(i+1)^{k+1}$.

Using similar arguments as on the first two pages and using property of summation:

$$\sum_{i=1}^{n}(i+1)^{k+1} = \sum_{i=2}^{n+1} i^{k+1} = \sum_{i=1}^{n} i^{k+1} + (n+1)^{k+1} - 1 \qquad (1)$$

To make the derivation look better lets define $S_j[n]$ to be $\sum_{i=1}^{n} i^j$.

Using Binomial Theorem (see next page for definition) on LHS of (1):

$$S_{k+1}[n] = \sum_{i=1}^{n}\left(\sum_{j=0}^{k+1}\left[\frac{(k+1)!}{(k+1-j)!\,j!} i^j\right]\right) \qquad (2)$$

Using the fact that we can interchange the summations:

$$S_{k+1}[n] = \sum_{j=0}^{k+1}\left(\sum_{i=1}^{n}\left[\frac{(k+1)!}{(k+1-j)!\,j!} i^j\right]\right) \qquad (3)$$

Since $\frac{(k+1)!}{(k+1-j)!\,j!}$ is constant with respect to ***i***, then we can pull it out from the inner sum:

$$S_{k+1}[n] = \sum_{j=0}^{k+1}\left(\frac{(k+1)!}{(k+1-j)!\,j!}\left[\sum_{i=1}^{n} i^j\right]\right) \qquad (4)$$

But $\left[\sum_{i=1}^{n} i^j\right]$, $S_j[n]$, is the sum formula for power ***j***; (4) becomes:

$$S_{k+1}[n] = \sum_{j=0}^{k+1}\left(\frac{(k+1)!}{(k+1-j)!\,j!} S_j[n]\right) \qquad (5)$$

But we are looking for $S_k[n]$ and since the formula (5) depends on $S_k[n]$ and $S_{k+1}[n]$ (look at the summation limit) we would like to pull them out of the sum; (5) becomes:

$$\sum_{j=0}^{k-1}\left(\frac{(k+1)!}{(k+1-j)!\,j!} S_j[n]\right) + \frac{(k+1)!}{(\cancel{k}+1-\cancel{k})!\,k!} S_k[n] + \frac{(k+1)!}{(\cancel{k}+\cancel{1}-\cancel{k}-\cancel{1})!\,(k+1)!} S_{k+1}[n]$$

$$= \sum_{j=0}^{k-1}\left(\frac{(k+1)!}{(k+1-j)!\,j!} S_j[n]\right) + \frac{(k+1)!}{k!} S_k[n] + \frac{(k+1)!}{(k+1)!} S_{k+1}[n]$$

$$= \sum_{j=0}^{k-1}\left(\frac{(k+1)!}{(k+1-j)!\,j!} S_j[n]\right) + (k+1)S_k[n] + S_{k+1}[n] = \sum_{i=1}^{n}(i+1)^{k+1} \qquad (6)$$

equating LHS of (6) and RHS of (1):

$$\sum_{j=0}^{k-1}\left(\frac{(k+1)!}{(k+1-j)!\,j!} S_j[n]\right) + (k+1)S_k[n] + S_{k+1}[n] = \sum_{i=1}^{n} i^{k+1} + (n+1)^{k+1} - 1 = S_{k+1}[n] + (n+1)^{k+1} - 1$$

rearranging:

$$(k+1)S_k[n] + \cancel{S_{k+1}[n]} = \cancel{S_{k+1}[n]} + (n+1)^{k+1} - 1 - \sum_{j=0}^{k-1}\left(\frac{(k+1)!}{(k+1-j)!\,j!} S_j[n]\right)$$

and finally dividing by (k+1) and using Binomial Theorem on $(n+1)^{k+1}$:

$$\boxed{\sum_{i=1}^{n} i^k = \frac{\sum_{l=0}^{k+1}\left(\frac{(k+1)!}{(k+1-l)!\,l!} n^l\right) - 1 - \sum_{j=0}^{k-1}\left(\frac{(k+1)!}{(k+1-j)!\,j!} S_j[n]\right)}{k+1}}$$

Binomial Theorem

Binomial Theorem gives a general polynomial expansion for $(a+b)^n$:

$$(a+b)^n=\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}a^{n-k}b^k$$

(n! is called factorial of n, for definition see the bottom of the page)
The theorem is usually proved using induction and the proof is not included here.

Examples of use of Binomial Theorem:

$$(a+b)^2=\frac{2!}{(2-0)!0!}a^{2-0}b^0+\frac{2!}{(2-1)!1!}a^{2-1}b^1+\frac{2!}{(2-2)!2!}a^{2-2}b^2=a^2+2ab+b^2$$

$$(a+b)^3=\frac{3!}{(3-0)!0!}a^{3-0}b^0+\frac{3!}{(3-1)!1!}a^{3-1}b^1+\frac{3!}{(3-2)!2!}a^{3-2}b^2+\frac{3!}{(3-3)!3!}a^{3-3}b^3=a^3+3a^2b+3ab^2+b^3$$

Factorials

n! is called a factorial of n and is defined as follows:
$n! =$ n×(n-1)×(n-2)×...×(1) for n>0

Example: $3!=3\cdot 2\cdot 1=6$

Some properties of factorials:
$n!=n\cdot[(n-1)!]$
1!=1
0!=1
(-x)!=0, where x>0